Math 206A Lecture 13 Notes

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1 Balinski's Theorem and Associahedra

1.1 Balinski's theorem

Definition 1.1. A graph $G = (V, E)$ is called k**-connected** if for every $(k-1)$ vertices $v_1, \dots, v_{k-1}$, $G \setminus \{v_1, \dots, v_{k-1}\}$ is connected.

Theorem 1.1 (Balinski)**.** *For every convex polytope $P \subseteq \mathbb{R}^d$ with $\dim(P) = d$, $\Gamma = \Gamma(P)$ is d-connected.*

For $d = 2$, the graph is a cycle, so removing a vertex does not disconnect the graph.

Proof. Suppose $X = \{v_1, \dots, v_{d-1}\} \subseteq V(P)$. Choose any vertex $z \in V \setminus X$. Let H be a hyperplane spanned by $X \cup \{z\}$. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a linear function such that $\psi(v_i) = \psi(z) = 0$, and let ψ' to be a small perturbation of ψ which is nonconstant on H. Let u be the vertex maximizing φ and w be the vertex minimizing φ. Also, let $H_- = \{x \in V : \psi'(x) < 0\}$ and $H_+ = \{y \in V : \psi'(y) > 0\}$.

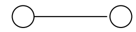
If we start at $y \in H_+$ and travel along edges where ψ' is increasing, we end up at u. If we start at $x \in H_-$ and travel along edges where ψ' is decreasing, we end up at w. So we know that H^+ and H_- are connected. We claim that z is connected to both u and w. Depending on our choice of perturbation φ, $\varphi(z) > 0$, in which case z is connected to H_+, or $\varphi(z) < 0$, in which case z is connected to H_-. ☐

1.2 Associahedra

Fix $n \geq 3$, and construct the graph $\Gamma = (V, E)$, where V is the set of triangulations of an n-gon ($|V| = \binom{2n}{n}/(n+1)$, the n-th Catalan number) and E is the set of triangulations that differ by a flip. Here, a flip means removing an edge in the triangulation and replacing it with the opposite diagonal of the resulting quadrilateral. Then Γ is $n-3$ regular because an n-gon has $n-3$ diagonals.

Is Γ the graph of a simple polytope in $\mathbb{R}^{n-3}$?

Example 1.1. For $n = 4$, we get



For $n = 5$, we get the graph of a pentagon. For $n = 6$, the graph has 14 vertices; try to come up with it yourself![1]

Theorem 1.2. *Let $\Gamma = (V, E)$ be the above graph. It is a graph of a simple polytope P_n.*

Stasheff said that $\alpha(P_n)$ is the set of subdivisions of the n-gon by non-crossing diagonals, ordered by inclusion. K. Lee showed that yes, there exists such a polytope P_n.

Here is the Gelfan-Zelevinsky-Kapranov construction.[2] For each triangulation T of a fixed n-gon Q, let $f_T : V(Q) \to \mathbb{R}_+$ be

$$f(v) = \sum_{\triangle \ni v} \mathbf{area}(\triangle)$$

Theorem 1.3 (GZK,c.1990)**.** *For every Q, the set of f_T for all triangulations of A is the set of vertices of the associahedron P_n; i.e. $P = \operatorname{conv}(\{f_T\})$.*

P_n sits in $\mathbb{R}^n$. What linear equations does it satisfy that makes the dimension $n - 3$? One equation is

$$\sum_{v \in V(Q)} f_T(v) = 3 \operatorname{area}(Q).$$

Theorem 1.4 (TTQ)**.** *For $n > 20$, $\operatorname{diam}(\Gamma_n) = 2n - 10$.*

Proving that $\operatorname{diam}(\Gamma_n) \geq 2n - 10$ is the easier part, but $\operatorname{diam}(\Gamma_n) \leq 2n - 10$ is hard.

Adelson-Velsky-Landis[3] trees: If you have a binary tree with too much depth on one side of the root, you might want to choose a different root so the tree is more balanced. This is related to triangulations of an n-gon because the dual graph of a triangulation is a binary tree.

[1] There's no way I'm making a diagram for this one.

[2] The names are in this order because alphabetic order in Russian is different from alphabetic order in English.

[3] Adelson-Velsky is one person.